PARTIAL DIFFERENTIAL EQUATIONS

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Formation of Partial Differential equations

Partial Differential Equation can be formed either by elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables .

SOLVED PROBLEMS

1.Eliminate two arbitrary constants a and b from $(x-a)^2+(y-b)^2+z^2=R^2$ here R is known constant .

(OR) Find the differential equation of all spheres of fixed radius having their centers in x y- plane.

solution

$$(x-a)^2+(y-b)^2+z^2=R^2 \ldots\ldots.(1)$$

Differentiating both sides with respect to x and y

$$2z\frac{\partial z}{\partial x}=-2(x-a)$$

$$2z\frac{\partial z}{\partial y}=-2(y-b)$$

$$\frac{\partial z}{\partial x}=p,\frac{\partial z}{\partial y}=q$$

$$\therefore\ x-a=-pz,\ y-b=-qz$$

By substituting all these values in (1)

$$p^2 z^2 + q^2 z^2 + z^2 = R^2$$

$$\Rightarrow z^2 = \frac{R^2}{p^2 + q^2 + 1}$$

or

$$z^2 = \frac{R^2}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

2. Find the partial Differential Equation by eliminating arbitrary functions from $z = f(x^2 - y^2)$

SOLUTION

$$z = f(x^2 - y^2) \ldots\ldots\ldots\ldots(1)$$

$$d.w.r.to.xandy$$

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2) \times 2x \ldots\ldots(2)$$

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2) \times -2y \ldots\ldots(3)$$

By $\frac{(2)}{(3)}$

$$\frac{\left(\frac{\partial z}{\partial x}\right)}{\left(\frac{\partial z}{\partial y}\right)} = \frac{-x}{y}$$

$$\frac{p}{q} = \frac{-x}{y} \Rightarrow py + qx = 0$$

3.Find Partial Differential Equation by eliminating two arbitrary functions from

$$z = yf(x) + xg(y)$$

SOLUTION

$$z = yf(x) + xg(y) \ldots\ldots(1)$$

Differentiating both sides with respect to x and y

$$\frac{\partial z}{\partial x} = yf'(x) + g(y) \ldots\ldots\ldots(2)$$

$$\frac{\partial z}{\partial y} = f(x) + xg'(y) \ldots\ldots\ldots(3)$$

Again d . w .r. to x and yin equation (2)and(3)

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

$$x \times (2) + y \times (3) ......to...get$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$$

$$xg(y)+yf(x)+xy(f'(x)+g'(y))$$

$$=z+xy\left(f'+g'\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z+xy\left(\frac{\partial^2 z}{\partial x\partial y}\right)$$

Different Integrals of Partial Differential Equation

1. Complete Integral (solution)

Let $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = F(x, y, z, p, q) = 0 \ldots\ldots(1)$

be the Partial Differential Equation.

The complete integral of equation (1) is given by

$$\phi(x, y, z, a, b) = 0 \ldots\ldots\ldots(2)$$

2. Particular solution

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called particular solution .

3.Singular solution

The eliminant of a , b between

$$\phi(x, y, z, a, b) = 0$$

$$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$$

when it exists , is called singular solution

4. General solution

In equation (2) assume an arbitrary relation of the form $b = f(a)$. Then (2) becomes

$$\phi(x, y, z, a, f(a)) = 0 \ldots\ldots\ldots (3)$$

Differentiating (2) with respect to a,

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} f'(a) = 0 \ldots\ldots\ldots (4)$$

The eliminant of (3) and (4) if exists, is called general solution

Standard types of first order equations

TYPE-I

The Partial Differential equation of the form

$$f(p,q)=0$$

has solution

$$z=ax+by+c \quad \text{with} \quad f(a,b)=0$$

TYPE-II

The Partial Differential Equation of the form

$z = px+qy+f(p,q)$ is called ***Clairaut's*** form of *pde* , it's solution is given by

$$z=ax+by+f(a,b)$$

TYPE-III

If the *pde* is given by $f(z, p, q) = 0$

then assume that

$$z = \phi(x + ay)$$

$$u = x + ay$$

$$z = \phi(u)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}.1 = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}.a = a\frac{dz}{du}$$

$\therefore$ The given *pde* can be written as

$f(z, \frac{dz}{dx}, a\frac{dz}{dy}) = 0$ And also this can

be integrated to get solution

TYPE-IV

The *pde* of the form $f(x,p) = g(y,q)$ can be solved by assuming

$$f(x,p) = g(y,q) = a$$

$$f(x,p) = a \Rightarrow p = \phi(x,a)$$

$$g(y,q) = a \Rightarrow q = \Psi(y,a)$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = \phi(x,a)dx + \Psi(y,a)dy$$

Integrate the above equation to get solution

SOLVED PROBLEMS

1. Solve the *pde* $p^2 - q = 1$ and find the complete and singular solutions

Solution

Complete solution is given by

$$z = ax + by + c$$

with $$a^2 - b = 1$$

$$\Rightarrow b = a^2 - 1$$

$$z = ax + (a^2 - 1)y + c$$

d.w.r.to. a and c then

$$\frac{\partial z}{\partial a} = x + 2ay$$

$$\frac{\partial z}{\partial c} = 1 = 0$$ Which is not possible

Hence there is no singular solution

2.Solve the *pde* $pq + p + q = 0$ and find the complete, general and singular solutions

Solution

The complete solution is given by

$$z = ax + by + c$$

with

$$ab + a + b = 0$$

$$a = \frac{-b}{b+1}$$

$$\therefore z = \frac{-b}{b+1}x + by + c \text{.......}(1)$$

$$\frac{\partial z}{\partial b} = \frac{-1}{(b+1)^2} x + y = 0$$

$$\frac{\partial z}{\partial c} = 1 = 0$$ no singular solution

To get general solution assume that

$$c = g(b)$$

From eq (1)

$$\therefore z = \frac{-b}{b+1} x + by + g(b) \ldots\ldots.(2)$$

$$\frac{\partial z}{\partial c} = \frac{-1}{(b+1)^2} x + y + g'(b) ....(3)$$

Eliminate from (2) and (3) to get general solution

3.Solve the *pde* $z = px + qy + \sqrt{1 + p^2 + q^2}$ and find the complete and singular solutions

Solution

The pde $z = px + qy + \sqrt{1 + p^2 + q^2}$ is in Clairaut's form complete solution of (1) is

$$z = ax + by + \sqrt{1 + a^2 + b^2} \text{.......}(2)$$

d.w.r.to "a" and "b"

$$\left.\begin{aligned} \frac{\partial z}{\partial a} &= x + \frac{a}{\sqrt{1 + a^2 + b^2}} = 0 \\ \frac{\partial z}{\partial b} &= y + \frac{b}{\sqrt{1 + a^2 + b^2}} = 0 \end{aligned}\right\} \text{........}(3)$$

From (3)

$$x^2 = \frac{a^2}{1+a^2+b^2}, y^2 = \frac{b^2}{1+a^2+b^2}$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$\Rightarrow \frac{1}{1+a^2+b^2} = 1-(x^2+y^2)$$

$$ax + \frac{a^2}{\sqrt{1 + a^2 + b^2}} = 0$$

$$by + \frac{b^2}{\sqrt{1 + a^2 + b^2}} = 0$$

$$ax + by + \sqrt{1 + a^2 + b^2} - \frac{1}{\sqrt{1 + a^2 + b^2}} = 0$$

$$z - \frac{1}{\sqrt{1 + a^2 + b^2}} = 0 \Rightarrow z^2 = 1 - (x^2 + y^2)$$

$\Rightarrow x^2 + y^2 + z^2 = 1$ is required singular solution

4. Solve the pde $(1-x)p+(2-y)q=3-z$

Solution

pde $$(1-x)p+(2-y)q=3-z$$

$$z = px+qy+(3-p-2q)$$

Complete solution of above pde is

$$z = ax+by+(3-a-2b)$$

5. Solve the pde $p^2+q^2=z$

Solution

Assume that $z=\phi(x+ay)$

$$u = x + ay$$

$$z = \phi(u)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}.1 = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}.a = a\frac{dz}{du}$$

From given pde

$$p^2 + q^2 = z \Rightarrow \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z}{1+a^2}$$

$$\left(\frac{dz}{du}\right) = \sqrt{\frac{z}{1+a^2}} \Rightarrow \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}du$$

Integrating on both sides

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + b$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + b$$

6. Solve the pde $zpq = p + q$

Solution

Assume $q = ap$

Substituting in given equation

$$zpap = p + ap$$

$$p = \frac{1+a}{az}, q = \frac{1+a}{z}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = \frac{1+a}{az} dx + \frac{1+a}{z} dy$$

$$zadz = (1+a)(dx + ady)$$

Integrating on both sides

$$\frac{a}{2}z^2 = (1+a)(x+ay)+b$$

7.Solve pde $pq = xy$

(or) $\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = xy$

Solution

$$\frac{p}{x} = \frac{q}{y}$$

Assume that

$$\frac{p}{x} = \frac{y}{q} = a$$

$$\therefore\ p = ax, q = \frac{y}{a}$$

$$dz = \ pdx + qdy = axdx + \frac{y}{a}dy$$

Integrating on both sides

$$z = a\frac{x^2}{2} + \frac{y^2}{2a} + b$$

8. Solve the equation $p^2+q^2=x+y$

Solution

$$p^2-x=y-q^2=a$$

$$p=\sqrt{a+x},q=\sqrt{y-a}$$

$$dz=pdx+qdy=\sqrt{a+x}dx+\sqrt{y-a}dy$$

integrating

$$z=\frac{2}{3}(a+x)^{\frac{3}{2}}+(y-a)^{\frac{3}{2}}+b$$

Equations reducible to the standard forms

(i)If $(x^m p)$ and $(y^n q)$ occur in the ***pde*** as in

$F(x^m p, y^n q) = 0$ Or in $F(z, x^m p, y^n q) = 0$

Case (a) Put $x^{1-m} = X$ and $y^{1-n} = Y$
if $m \neq 1$; $n \neq 1$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)x^{-m}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial Y}(1-n)y^{-n}$$

$$x^m p = \frac{\partial z}{\partial X}(1-m) = P(1-m)$$

$$y^n q = \frac{\partial z}{\partial Y}(1-n) = Q(1-n)$$

where $$\frac{\partial z}{\partial X} = P, \frac{\partial z}{\partial Y} = Q$$

Then $F(x^m p, y^n q) = 0$ reduces to $F(P, Q) = 0$

Similarly $F(z, x^m p, y^n q) = 0$ reduces to $F(z, P, Q) = 0$

case(b)

If $m=1$ or $n=1$

put $\log x = X, \log y = Y$

$$p = \frac{\partial z}{\partial X}\frac{1}{x} \Rightarrow px = P$$

$$q = \frac{\partial z}{\partial Y}\frac{1}{y} \Rightarrow qy = Q$$

(ii)If $(z^k p)$ and $(z^k q)$ occur in ***pde*** as in $F(z^k p, z^k q)$

Or in $f_1(x, z^k p) = f_2(y, z^k q)$

Case(a) Put $z^{1+k} = Z$ if $k \neq -1$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = z^{-k}(1+k)^{-1}\frac{\partial Z}{\partial x} \Rightarrow z^{k} p = (1+k)^{-1} P$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y} = z^{-k}(1+k)^{-1}\frac{\partial Z}{\partial y} \Rightarrow z^{k} q = (1+k)^{-1} Q$$

where $\frac{\partial Z}{\partial x} = P, \frac{\partial Z}{\partial y} = Q$

Given pde reduces to

$F(P,Q)$ and $f_1(x,P) = f_2(y,Q)$

Case(b) if $k = -1$ $\quad \log z = Z$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \Rightarrow z^{-1}p = P$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y} = z\frac{\partial Z}{\partial y} \Rightarrow z^{-1}q = Q$$

Solved Problems

1.Solve $p^2x^4 + q^2y^4 = z^2$

Solution $\left(\frac{px^2}{z}\right)^2 + \left(\frac{qy^2}{z}\right)^2 = 1.......(1)$

$$m = 2, n = 2$$

$$k = -1$$

$$x^{-1} = X \qquad y^{-1} = Y \qquad \log z = Z$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial X}\frac{\partial X}{\partial x} = -zx^{-2}\frac{\partial Z}{\partial X} = -zx^{-2}P$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y} = -zy^{-2}\frac{\partial Z}{\partial Y} = -zy^{-2}Q$$

where $\frac{\partial Z}{\partial X} = P, \frac{\partial Z}{\partial Y} = Q$

$$\frac{px^2}{z} = -P, \frac{qy^2}{z} = -Q$$

(1)becomes

$$(-P)^2 + (-Q)^2 = 1$$

$$P^2 + Q^2 = 1$$

$$\therefore Z = aX + bY + c$$

$$a^2 + b^2 = 1, b = \sqrt{1-a^2}$$

$$\log z = ax^2 + \sqrt{1-a^2}\, y^2 + c$$

2. Solve the pde $p^2 + q^2 = z^2(x^2 + y^2)$

SOLUTION

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = (x^2 + y^2).....(1)$$

$$k = -1 \quad \log z = Z$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = z\frac{\partial Z}{\partial x} \Rightarrow z^{-1}p = P$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial y} = z\frac{\partial Z}{\partial y} \Rightarrow z^{-1}q = Q$$

Eq(1) becomes

$$P^2 + Q^2 = (x^2 + y^2).....(2)$$

$$P^2 - x^2 = y^2 - Q^2 = a^2$$

$$\log z = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}(a^2 + x^2) +$$

$$\frac{y\sqrt{(y^2 - a^2)}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b$$

Lagrange's Linear Equation

Def: The linear partial differenfial equation of first order is called as Lagrange's linear Equation.

This eq is of the form $Pp + Qq = R$

Where P, Q and R are functions x,y and z

The general solution of the partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$

Where F is arbitrary function of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

Here $u = c_1$ and $v = c_2$ are independent solutions of the auxilary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Solved problems

1.Find the general solution of $x^2 p + y^2 q = (x+y)z$

Solution

auxilary equations are $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$ Integrating on both sides

$$u = \left(x^{-1} - y^{-1}\right) = c_1$$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x + y)z}$$

$$\frac{d(x - y)}{(x - y)(x + y)} = \frac{dz}{(x + y)z}$$

$$\frac{d(x - y)}{(x - y)} = \frac{dz}{z}$$ Integrating on both sides

$$\log(x-y)=\log z+\log c_2$$

$$v=(x-y)z^{-1}=c_2$$

The general solution is given by $F(u,v)=0$

$$F(x^{-1}-y^{-1},(x-y)z^{-1})=0$$

2.solve $x^2(y-z)+y^2(z-x)q=z^2(x-y)$

solution

Auxiliary equations are given by

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

$$\frac{\frac{dx}{x^2}}{(y-z)} = \frac{\frac{dy}{y^2}}{(z-x)} = \frac{\frac{dz}{z^2}}{(x-y)}$$

$$\frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{(y-z)+(z-x)+(x-y)}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating on both sides

$$u = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$$

$$\frac{x^{-1}dx}{x(y-z)} = \frac{y^{-1}dy}{y(z-x)} = \frac{z^{-1}dz}{z(x-y)}$$

$$\frac{x^{-1}dx + y^{-1}dy + z^{-1}dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$ Integrating on both sides

$$v = xyz = b$$

The general solution is given by

$$F(x^{-1}+y^{-1}+z^{-1}, xyz)=0$$

HOMOGENEOUS LINEAR PDE WITH CONSTANT COEFFICIENTS

Equations in which partial derivatives occurring are all of same order (with degree one) and the coefficients are constants ,such equations are called homogeneous linear PDE with constant coefficient

$$\frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \ldots\ldots\ldots a_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Assume that $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}.$

then n^{th} order linear homogeneous equation is given by

$$(D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \ldots\ldots\ldots + a_n D'^n) z = F(x, y)$$

or

$$f(D, D') z = F(x, y) \ldots\ldots\ldots (1)$$

The complete solution of equation (1) consists of two parts ,the complementary function and particular integral.

The complementary function is complete solution of equation of $f(D, D')z = 0$

Rules to find complementary function

Consider the equation

$$\frac{\partial^2 z}{\partial x^2} + k_1 \frac{\partial^2 z}{\partial x \partial y} + k_2 \frac{\partial^2 z}{\partial y^2} = 0$$

or

$$(D^2 + k_1 DD' + k_2 D'^2)z = 0 \ldots\ldots\ldots\ldots(2)$$

The auxiliary equation for (A.E) is given by

$$D^2 + k_1 DD' + k_2 D'^2 = 0$$

And by giving $D = m, D' = 1$

The A.E becomes $m^2 + k_1 m + k_2 = 0 \ldots (3)$

Case 1

If the equation(3) has two distinct roots m_1, m_2

The complete solution of (2) is given by

$$z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

Case 2

If the equation(3) has two equal roots i.e $m_1 = m_2$

The complete solution of (2) is given by

$$z = f_1(y + m_1 x) + x f_2(y + m_1 x)$$

Rules to find the particular Integral

Consider the equation

$$(D^2 + k_1 DD' + k_2 D'^2) z = F(x, y)$$

$$f(D, D') z = F(x, y)$$

Particular Integral (P.I) $= \dfrac{F(x,y)}{f(D,D')}$

Case 1 If $F(x,y) = e^{ax+by}$

then P.I= $\dfrac{1}{f(D,D')} e^{ax+by}$

$$= \frac{1}{f(a,b)} e^{ax+by}, \; f(a,b) \neq 0$$

If $f(a,b) = 0$ and $(D - \frac{a}{b} D')$ is

factor of $f(D,D')$ then

P.I $= xe^{ax+by}$

If $f(a,b)=0$ and $(D-\frac{a}{b}D')^2$ is factor of $f(D,D')$

then P.I $= \frac{x^2}{2}e^{ax+by}$

Case 2

$$F(x,y)=\sin(mx+ny) or \cos(mx+ny)$$

P.I $= \frac{\sin(mx+ny)}{f(D^2,DD',D'^2)} = \frac{\sin(mx+ny)}{f(-m^2,-mn,-n^2)}$

Case 3 $F(x,y) = x^m y^n$

P.I $= \frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operating on $x^m y^n$ term by term.

Case 4 when $F(x,y)$ is any function of x and y.

P.I= $\frac{1}{f(D,D')} F(x,y)$

$$\frac{1}{D - mD'} F(x,y) = \int F(x, c - mx)\,dx$$

Here $(D - mD')$ is factor of $f(D, D')$

Where ‘c’ is replaced by $(y + mx)$ after integration

Solved problems

1.Find the solution of ***pde***

$$(D^3 - D'^3 + 3DD'^2 - 3D^2D')z = 0$$

Solution

The Auxiliary equation is given by

Solution

The Auxiliary equation is given by

$$m^3 - 1 + 3m - 3m^2 = 0$$

By taking $D = m, D' = 1$

$$\therefore m = 1,1,1.$$

Complete solution $= f_1(y+x) + xf_2(y+x) + x^2 f_3(y+x)$

2. Solve the ***pde*** $(D^3 + 4D^2D' - 5DD')z = 0$

Solution

The Auxiliary equation is given by

$$m^3 + 4m^2 - 5m = 0$$

$$\therefore m = 0, 1, -5$$

$$z = f_1(y) + f_2(y + x) + f_3(y - 5x)$$

3. Solve the ***pde*** $(D^2 + D'^2)z = 0$

Solution

the A.E is given by $m^2 + 1 = 0$

$$m = \pm i$$

$$\therefore z = f_1(y + ix) + f_2(y - ix)$$

4. Find the solution of ***pde***

$$(D^2 + 3DD' - 4D'^2)z = e^{2x+4y}$$

Solution

Complete solution =
Complementary Function + Particular Integral

The A.E is given by $m^2 + 3m - 4 = 0$

$$\boldsymbol{m = -4,1}$$

$$C.F = \phi_1(y + x) + \phi_2(y - 4x)$$

$$P.I = \frac{e^{2x+4y}}{D^2 + 3DD' - 4D'^2} = \frac{e^{2x+4y}}{-36}$$

Complete solution

$$= C.F + P.I$$

$$= \phi_1(y+x) + \phi_2(y-4x) - \frac{e^{2x+4y}}{36}$$

5.Solve $(D^3 - 3DD' + 2D'^3)z = e^{2x-y} + e^{x+y}$

Solution

$$A.E = m^3 - 3m + 2$$

$$\therefore m = 1, 1, -2.$$

$$C.F = \phi_1(y+x) + x\phi_2(y+x) + \phi_3(y-2x)$$

$$P.I_1 = \frac{e^{2x-y}}{D^3 - 3DD'^2 + 2D'^3} = \frac{e^{2x-y}}{(D-D')^2(D^2+2D')}$$

$$P.I_1 = \frac{e^{2x-y}}{(D-D')^2(D+2D')} = \frac{xe^{2x-y}}{9}$$

$$P.I_2 = \frac{e^{x+y}}{D^3 - 3DD'^2 + 2D'^3} = \frac{e^{x+y}}{(D-D')^2(D^2+2D')}$$

$$\therefore P.I_2 = \frac{x^2}{6}e^{x+y}$$

$$z = C.F + P.I_1 + P.I_2$$

$$z=\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y-2x)+\frac{xe^{2x-y}}{9}+\frac{x^2}{6}e^{x+y}$$

6.Solve $(D^2-DD')z=\cos x\cos 2y$

Solution

$$(D^2-DD')z=\frac{1}{2}\left[\cos(x+2y)+\cos(x-2y)\right]$$

$$A.E=m^2-m=0$$

$$m=0,1$$

$$C.F=\phi_1(y+x)+\phi_2(y)$$

$$P.I_1 = \frac{\cos(x+2y)}{(D^2 - DD')} = \frac{\cos(x+2y)}{((-1)-(-2))} = \cos(x+2y)$$

$$P.I_2 = \frac{\cos(x-2y)}{(D^2 - DD')} = \frac{\cos(x-2y)}{((-1)-(2))} = \frac{\cos(x-2y)}{-3}$$

$$z = \phi_1(y+x) + \phi_2(y-x) + \cos(x+2y) - \frac{1}{3}\cos(x-2y)$$

7.Solve $(D^2 + DD' - 6D'^2)z = x^2 y^2$

Solution $A.E = m^2 + m - 6 = 0$

$m = 2, -3.$

$$C.F = \phi_1(y+2x) + \phi_2(y-3x)$$

$$P.I = \frac{x^2 y^2}{D^2 + DD' - 6D'^2}$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right]^{-1} x^2 y^2$$

$$= D^{-2}\left[1 - \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right) + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)^2\right] x^2 y^2$$

$$= D^{-2}\left[1-\left(\frac{D'}{D}-6\frac{D'^2}{D^2}\right)+\frac{D'^2}{D^2}\right]x^2y^2$$

$$= D^{-2}\left[x^2y^2-\left(\frac{2x^2y}{D}-6\frac{2x^2}{D^2}\right)+\frac{2x^2}{D^2}\right]$$

$$= D^{-2}\left[x^2y^2-\left(\frac{2x^3y}{3}-6\frac{2x^4}{12}\right)+\frac{2x^4}{12}\right]$$

$$= D^{-2}\left[x^2y^2+\frac{2x^3y}{3}+8\frac{2x^4}{12}\right]$$

$$=\left[\frac{x^4y^2}{12}+\frac{2x^5y}{60}+\frac{2x^6}{90}\right]$$

7.Solve $(D^2 - 5DD' + 6D'^2)z = y\sin x$

Solution

A.E is $m^2 - 5m + 6 = 0$

$$m = 3, m = 2.$$

$$C.F = \phi_1(y+3x) + \phi_2(y+2x)$$

$$P.I = \frac{y\sin x}{D^2 - 5DD' + 6D'^2} = \frac{y\sin x}{(D-3D')(D-2D')}$$

$$= \frac{1}{(D-3D')}\left[\frac{y\sin x}{(D-2D')}\right]$$

$$= \frac{1}{(D-3D')}\int (a-2x)\sin x\,dx$$

$$= \frac{1}{(D-3D')}\left[-a\cos x - 2(-x\cos x + \sin x)\right]$$

$$= \frac{1}{(D-3D')}\left[2x\cos x - 2\sin x - (y+2x)\cos x\right]$$

$$P.I = \frac{y\sin x}{D^2 - 5DD' + 6D'^2} = \frac{y\sin x}{(D-3D')(D-2D')}$$

$$= \frac{1}{(D-3D')}\left[\frac{y\sin x}{(D-2D')}\right]$$

$$= \frac{1}{(D-3D')}\int (a-2x)\sin x\,dx$$

here

$$(a = y + 2x)$$

$$= \frac{1}{(D-3D')}\left[-a\cos x - 2(-x\cos x + \sin x)\right]$$

$$= \frac{1}{(D-3D')}\left[2x\cos x - 2\sin x - (y+2x)\cos x\right]$$

$$= \frac{1}{(D-3D')}\left[-y\cos x - 2\sin x\right]$$

$$= \int(-(b-3x)\cos x - 2\sin x)dx$$

here

$$(b = y + 3x)$$

$$= -b\sin x + 2\cos x + 3(x\sin x + \cos x)$$

$$= -(y+3x)\sin x + 2\cos x + 3(x\sin x + \cos x)$$

$$= 5\cos x - y\sin x$$

Non Homogeneous Linear PDES

If in the equation $f(D, D')z = F(x, y)$..............(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non- homogeneous linear partial differential equation

Ex $$(D^2 + 3D + D' - 4D'^2)z = e^{2x+3y}$$

Complete Solution
= Complementary Function + Particular Integral

To find C.F., factorize $f(D, D')$
into factors of the form $(D - mD' - c)$

If the non homogeneous equation is of the form

$$(D - m_1 D' - c_1)(D - m_2 D' - c_2)z = F(x, y)$$

$$C.F = e^{c_1 x}\phi(y + m_1 x) + e^{c_2 x}\phi(y + m_2 x)$$

1.Solve $(D^2 - DD' + D)z = x^2$

Solution

$$f(D, D') = D^2 - DD' + D = D(D - D' + 1)$$

$$C.F = e^{-x}\phi_1(y + x) + \phi_2(y)$$

$$P.I = \frac{x^2}{D^2 - DD' + D} = \frac{1}{D^2}\left[1 - \frac{(D'+1)}{D}\right]^{-1} x^2$$

$$= \frac{1}{D^2}\left[x^2 + \left[\frac{(D'+1)}{D}\right]x^2 + \left[\frac{(D'+1)}{D}\right]^2 x^2 + \ldots\ldots\right]$$

$$= \frac{1}{D^2}\left[x^2 + \frac{x^3}{3} + \frac{x^4}{12}\right] = \left[\frac{x^4}{3.4} + \frac{x^5}{3.4.5} + \frac{x^6}{12.5.6}\right]$$

2.Solve $(D+D'-1)(D+2D'-3)z=4$

Solution

$$z=e^{x}\phi_1(y-x)+e^{3x}\phi_1(y-2x)+\frac{4}{3}$$